

A diffeomorphism anomaly in every dimension

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Abstract

Field-theoretic pure gravitational anomalies only exist in $4k + 2$ dimensions. However, canonical quantization of non-field-theoretic systems may give rise to diffeomorphism anomalies in any number of dimensions. I present a simple example, where a higher-dimensional generalization of the Virasoro algebra arises upon quantization.

The Polyakov action in string theory is nothing but 1+1-dimensional gravity coupled to scalar fields [2]. It thus seems reasonable to expect that canonical quantization of 3+1-dimensional gravity should proceed along similar lines. We know from string theory that quantization of the coordinate fields gives rise to a conformal anomaly, eventually to be cancelled by ghosts. Analogously, one may expect that quantization of the fields alone gives rise to an anomaly in the diffeomorphism constraint.

At first sight, this idea appears to be ruled out by two no-go theorems:

1. The diffeomorphism algebra in N dimensions has no central extension, except when $N = 1$ [7].
2. There are no pure gravitational anomalies in 3+1 dimensions.

However, these no-go theorems can be evaded. The extension (13) that generalizes the Virasoro algebra to several dimensions is not a central one, except in one dimension. In general, it is an extension of $\mathfrak{vect}(N)$, the algebra of vector fields (diffeomorphism algebra) in N dimensions, by its modules of closed $(N - 1)$ -forms. A closed zero-form is a constant function is the trivial module, so the Virasoro extension is central when $N = 1$, but not otherwise.

The claim that there are no diffeomorphism anomalies in four dimensions is simply false. What is true and has been proven is that there are no *field-theoretic* diffeomorphism anomalies [1]. Non-field-theoretic diffeomorphism

anomalies can and do arise. In this note I present a simple system which gives rise to an extension of the diffeomorphism algebra in N dimensions upon canonical quantization.

Consider a $2N$ -dimensional phase space with coordinates q^i and p_i , $i = 1, 2, \dots, N$, and Poisson brackets

$$\{q^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = \{q^i, q^j\} = 0. \quad (1)$$

$\mathbf{vect}(N)$ is generated by vector fields of the form $\xi = \xi^i(x)\partial_i$, where $\partial_i = \partial/\partial x^i$. It evidently acts on the space of functions of this phase space. Upon canonical quantization, the Poisson brackets are replaced by commutators

$$[q^i, p_j] = i\delta_j^i, \quad [p_i, p_j] = [q^i, q^j] = 0. \quad (2)$$

q^i and $p_j = -i\partial/\partial q^j$ become operators acting on the space of functions depending on the q 's alone, and $\mathbf{vect}(N)$ acts on this space as the operator $\mathcal{L}_\xi = i\xi^i(q)p_i$. This expression is already normal ordered, and there is no anomaly. In fact, it is easy to verify that $[\mathcal{L}_\xi, \mathcal{L}_\eta] = \mathcal{L}_{[\xi, \eta]}$, where $[\xi, \eta] = \xi^i\partial_i\eta^j\partial_j - \eta^j\partial_j\xi^i\partial_i$.

There is a simple generalization of this system in which $\mathbf{vect}(N)$ does acquire an extension. Consider the infinite-dimensional phase space with coordinates $q^i(s)$ and $p_i(s)$, where $s \in S^1$ is a parameter on a circle. We quantize this system by promoting $q^i(s)$ and $p_i(s)$ to operators satisfying the canonical commutator relations

$$[q^i(s), p_j(s')] = i\delta_j^i\delta(s - s'), \quad [p_i(s), p_j(s')] = [q^i(s), q^j(s')] = 0. \quad (3)$$

The Hilbert space is some space of functionals of half of the oscillators. One possibility is to consider the space of functionals of $q^i(s)$, with $p_i(s) = -i\delta/\delta q^i(s)$. In other words, we define a vacuum $|\text{vac}\rangle$ by $p_i(s)|\text{vac}\rangle = 0$, so $q^i(s)$ are creation operators and $p_i(s)$ are annihilation operators. $\mathbf{vect}(N)$ acts on this space without anomaly.

However, there is no Stone-von Neumann theorem in infinite dimension, so the choice of vacuum is important. A different choice does lead to an anomaly. Expand the phase space functions in a Fourier series in s , i.e.

$$\begin{aligned} q^i(s) &= \sum_{k=-\infty}^{\infty} \hat{q}^i(k)e^{-iks} \equiv q_{<}^i(s) + \hat{q}^i(0) + q_{>}^i(s), \\ p_i(s) &= \sum_{k=-\infty}^{\infty} \hat{p}_i(k)e^{-iks} \equiv p_i^{<}(s) + \hat{p}_i(0) + p_i^{>}(s). \end{aligned} \quad (4)$$

where $q_{<}^i(s)$ is the sum over negative Fourier ($k < 0$) modes alone, etc. Now define the new vacuum $|0\rangle$ by requiring that $q_{<}^i(s)$ and $p_i^{\leq}(s) = p_i^{<}(s) + \hat{p}_i(0)$ annihilate it. Normal ordering is defined as usual by moving the annihilation operators to the right. The normal-ordered $\mathbf{vect}(N)$ generators read

$$\begin{aligned}\mathcal{L}_\xi &= i \int ds : \xi^i(q(s)) p_i(s) : \\ &\equiv i \int ds (\xi^i(q(s)) p_i^{\leq}(s) + p_i^{>}(s) \xi^i(q(s))).\end{aligned}\tag{5}$$

By a straightforward calculation one finds that $\mathbf{vect}(N)$ has acquired an extension which generalizes the Virasoro algebra to N dimensions [4]:

$$\begin{aligned}[\mathcal{L}_\xi, \mathcal{L}_\eta] &= \mathcal{L}_{[\xi, \eta]} + \frac{1}{2\pi i} \int ds \frac{dq^k(s)}{ds} \left\{ c_1 \partial_k \partial_j \xi^i(q(s)) \partial_i \eta^j(q(s)) + \right. \\ &\quad \left. + c_2 \partial_k \partial_i \xi^i(q(s)) \partial_j \eta^j(q(s)) \right\},\end{aligned}\tag{6}$$

where $c_1 = 1$ and $c_2 = 0$. The classical system defined by (3) has thus acquired a diffeomorphism anomaly upon quantization.

This construction of anomalous $\mathbf{vect}(N)$ representations, which was first discovered in [6], can be considerably generalized. In fact, one can construct representations labelled by a $gl(N)$ representation and a positive integer [4]. The constant $c_2 \neq 0$ for some of these systems.

We now proceed to construct a Hamiltonian which commutes with (6).

$$\begin{aligned}H &= \int ds : \frac{dq^i(s)}{ds} p_i(s) : \\ &\equiv \int ds \left(\frac{dq^i(s)}{ds} p_i^{\leq}(s) + p_i^{>}(s) \frac{dq^i(s)}{ds} \right)\end{aligned}\tag{7}$$

is bounded from below, because the creation operators have positive eigenvalues:

$$[H, \hat{q}^i(k)] = k \hat{q}^i(k), \quad [H, \hat{p}_i(k)] = k \hat{p}_i(k).\tag{8}$$

Moreover, $[\mathcal{L}_\xi, H] = 0$, so the anomalous diffeomorphism algebra (6) is a symmetry of this Hamiltonian.

The equations of motion are somewhat pathological. The equations of motion for $q^i(s, t)$ and $p_j(s, t)$ read

$$\frac{\partial q^i}{\partial t} = \frac{\partial q^i}{\partial s}, \quad \frac{\partial p_j}{\partial t} = \frac{\partial p_j}{\partial s},\tag{9}$$

with the solution $q^i(s, t) = q^i(s + t)$, $p_j(s, t) = p_j(s + t)$. In particular, the non-zero Poisson brackets at non-equal times become

$$\{q^i(s, t), p_j(s', t')\} = \delta_j^i \delta(s + t - s' - t'). \quad (10)$$

We note that the Hamiltonian can be embedded into the family of operators

$$H_k = \int ds \, e^{iks} : \frac{dq^i(s)}{ds} p_i(s) :, \quad (11)$$

which generate an additional Virasoro algebra. Classically, this algebra commutes with $\mathbf{vect}(N)$, but normal ordering introduces an anomaly [4]. However, \mathcal{L}_ξ still commutes with the Hamiltonian $H = H_0$. We obtain a lowest-weight representation of this extra Virasoro algebra, since $H_k|0\rangle = 0$ for all $k < 0$.

To make the analogy with the Virasoro algebra very explicit, it is instructive to use a Fourier basis on the N -dimensional torus with coordinates $x = (x^i)$. Let $L_i(m)$ be the generator \mathcal{L}_ξ corresponding to the vector field $\xi = -i \exp(im_k x^k) \partial_i$, where $m = (m_i) \in \mathbb{Z}^N$ labels momenta. Moreover, set

$$S^i(m) = \frac{1}{2\pi i} \int ds \, \frac{dq^i(s)}{ds} \exp(im_k q^k(s)). \quad (12)$$

The multi-dimensional Virasoro algebra (6) takes the form [3, 6]

$$\begin{aligned} [L_i(m), L_j(n)] &= n_i L_j(m + n) - m_j L_i(m + n) \\ &\quad + (c_1 m_j n_i + c_2 m_i n_j) m_k S^k(m + n), \\ [L_i(m), S^j(n)] &= n_i S^j(m + n) + \delta_i^j m_k S^k(m + n), \\ [S^i(m), S^j(n)] &= 0, \\ m_i S^i(m) &= 0. \end{aligned} \quad (13)$$

It is easy to see that (13) generalizes the Virasoro algebra to higher dimensions. Namely, in one dimension we can ignore all indices, and the unique solution of the closedness condition $mS(m) = 0$ is proportional to a Kronecker delta, $S(m) \propto \delta(m)$. If we make this substitution into the other equations of (13), we immediately recover the defining relations for the Virasoro algebras, with the trivial linear cocycle put to zero. In particular, $[L(m), S(n)] = 0$.

The diffeomorphism anomalies described in this note are very different from field- and string-theoretical gravitational anomalies. In string theory,

diffeomorphism anomalies arise from chiral fermions and only exist if space-time has $4k + 2$ dimensions ([2], part II, page 336). Such anomalies, like pure gauge anomalies in the standard model, are very problematic from a mathematical point of view; in the Hamiltonian formalism, they give rise to Mickelsson-Faddeev algebras which apparently lack good representations [5]. In contrast, in this note we have constructed diffeomorphism anomalies that are similar to the conformal anomaly in the bosonic string. The quantization of the fields alone (without ghosts) gives rise to lowest-weight representations of an extension of the classical symmetry algebra (the Virasoro algebra in one or several dimensions), which has nothing to do with chiral fermions.

This expectation can be made mathematically rigorous. It is well known that all non-trivial unitary lowest-weight representations of the Virasoro algebra have a positive value of the central charge (discrete unitary series and $c \geq 1$). A unitary representation in higher dimensions must give rise to a unitary representation of each Virasoro subalgebra generated by vector fields depending on a single variable. If the representation of the higher-dimensional Virasoro algebra is non-trivial, at least one such subalgebra must be non-trivially represented, i.e. there must be an anomaly. This proves that if quantization of 3+1-dimensional gravity involves non-trivial, unitary, lowest-weight representations of the diffeomorphism constraint, as one may expect on general grounds, a diffeomorphism anomaly of the kind described above must arise.

Thiemann has recently quantized the Nambu-Goto string using methods from Loop Quantum Gravity (LQG) [8]. This construction has been criticized on the grounds that the conventional conformal anomaly does not appear, because the representations in LQG, albeit unitary, are not of lowest-weight type. More precisely, one can argue that LQG is not canonical quantization in the conventional sense, which should give rise to a conformal anomaly in the bosonic string and to a diffeomorphism anomaly in 3+1-dimensional gravity, eventually to be cancelled by ghosts. What has not been widely appreciated is that the same critique applies to string theory, which is also incapable of producing the necessary diffeomorphism anomaly, at least in four dimensions.

To conclude, I have described a classical canonical system whose Hamiltonian commutes with diffeomorphisms, and where the diffeomorphism algebra acquires non-trivial quantum corrections. One of the main open problems is how to define unitarity; since there is no invariant inner product, only dual spaces can be invariantly paired. Nevertheless, the fact that diffeomorphism anomalies in every dimension are possible outside a field-theoretical

framework is quite striking and apparently not widely known.

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